Marx’s Original 6-Sector Model of Monetary Circuit

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I. Introduction

As is well known, Marx left a total of eight manuscripts for *Capital*, Volume II, the second (Manuskript II or Ms. II in 1868 - 1870) and the eighth (Manuskript VIII or Ms. VIII in 1880 - 1881) of which were used by Engels for the edition of Part 3 of Volume II, i.e. the chapters on the so-called Reproduction Schemes. While Ms. VIII was fully used, 36 pages of Ms. II which were dedicated to a specific topic were systematically ignored by Engels. The topic treated in this omitted part was the 6-sector model of production. Marx's multi-sectoral model is widely known as the 2-sector model. However, this is true only for the Engels' edition of *Capital*, and Marx himself tackled in his original *Capital* manuscript with the 6-sector analysis, whose theoretical relevance cannot be reduced to that of his 2-sector analysis.

There are two main issues to investigate in Marx's 6-sector model: First, the induction of equilibrium conditions for output and demand in the 6-sector economy including workers' and capitalists' household, and the comparative statics with regard to the rate of surplus value as a varying parameter. Second, the specification of the routes of monetary circuit which mediates inter- or intra-sectoral commodity transactions, and the exemplification of the so-called law of monetary reflux. As the first issue is going to be closely treated in another paper by the author, this paper concentrates on the second issue.

In this omitted part of Ms. II, the industry is assumed to be vertically divided into three sectors for each of works' and capitalists' consumption goods as follows[1]:

Sector A3: sector for workers' consumption goods

Sector A2: sector for means of production for A3 and itself

Sector A1: sector for means of production for A2 and itself

Sector B3: sector for capitalists7 consumption goods

Sector B2: sector for means of production for B3 and itself

Sector B1: sector for means of production for B2 and itself

Besides the above 6 industry sectors, the economy includes workers' household and capitalists' household. The output value consists of constant capital, variable capital and surplus value, and the income of workers and capitalists stems exclusively from variable capital and surplus value respectively, and each class does not save, so that the simple reproduction (stationary state) is assumed. The price is assumed to be proportional to the labour value.

This paper is based on the earlier version published in Japanese in Mori (2009). In the following, Section 2 presents essential results of Marx's study of monetary circuit in the 6-sector model, and Section 5 evaluates their theoretical implications before Section 6 concludes the paper.

2. Marx's model of monetary circuit

2.1. Illustration of the law of monetary reflux

Marx exemplified the equilibrium state of the above characterized 6-sector economy by the following numerical example (see Table 1).

Table 1. Marx's example of 6-sector equilibrium

	A1	A2	A3	B1	B2	B3	workers	capitalists	total demand
A1	200	100	0	0	0	0	0	0	300
A2	0	100	200	0	0	0	0	0	300
A3	0	0	0	0	0	0	300	0	300
B1	0	0	0	200	100	0	0	0	300
B2	0	0	0	0	100	200	0	0	300
B3	0	0	0	0	0	0	0	300	300
wage	50	50	50	50	50	50			300
surplus value	50	50	50	50	50	50			300
output	300	300	300	300	300	300	300	300	

The table can be read in the same manner as a usual I-O table. Each column in the intermediate-demand area represents an industry sector showing the price components of its output. For example, Sector A3 produces output of the price (value) of 300 by using A2 of 200 and by paying 50 as wage to workers and 50 as surplus value to capitalists. Each column in the final-demand area represents each the workers' or capitalists' household showing the price components of its consumption. For example, workers' household consume A3 of 300.

In the omitted part of Manuskript II, Marx investigated the so-called law of monetary reflux based on the 6-sector model of production where the equilibrium conditions are assumed to be satisfied according to the numerical example of Table 1. For this purpose, he divided, first, the sales transactions in the economy into 12 independent segments called each 'partial circulation (teilweise Zirculation)', and second, he specified for each segment a specific route of monetary circulation to finally confirm the reflux of money to its starting point. Marx regarded each segment as independent in that the transactions can be completed within the segment and without any intersection with another segment. One could sort Marx's 12 segments to three groups according to their starting point of money.

The first group could be called 'wage reflux'-type of partial circulation. Each of six segments of transactions which are classified into this type is mediated by money that is initially paid by capitalists as wage to workers and returns to its starting point in the end. Marx then specified the route of monetary circulation for each segment in the manner below. To explain the notation, e.g. 50 (*i*|*j*) denotes 50 units of money to be paid by sector *i* to purchase good *j* where in the case of *j* = 'wage' and 'surplus value', the money is to be paid as wage to workers and as surplus value to capitalists respectively. The arrow denotes changing hands of money so that the number of arrows indicates the number of times the money circulates.

1) 50 (A3|wage)→50 (worker|A3)→50 (A3|wage)[2]
2) 50 (A2|wage)→50 (worker|A3)→50 (A3|A2)→ 50 (A2|wage)[3]
3) 50 (A1|wage)→50 (worker|A3) →50 (A3|A2) →50 (A2|A1)→50 (A1|wage)[4]
4) 50 (B3|wage)→50 (worker|A3)→50 (A3|surplus value) / 50 (capitalist|B3)→50 (B3|wage)[5]
5) 50 (B2|wage)→50 (worker|A3)→50 (A3|A2)→50 (A2|surplus value) / 50 (capitalist|B3)→50 (B3|B2)→50 (B2|wage)[6]
6) 50 (B1|wage)→50 (worker|A3)→50 (A3|A2)→50 (A2|A1)→50 (A1|surplus value) / 50 (capitalist|B3)→50 (B3|B2)→50 (B2|B1)→50 (B1|wage)[7]

The second group is a set of those segments which could be characterized as 'capitalist expenditure'-type of circulation. Classified into this group are two segments of transactions which the money expended by capitalists for their individual consumption mediates until it returns to its starting point in the end. The route of monetary circulation in these segments is specified by Marx as follows:

7) 25 (B3|surplus value) / 25 (capitalist|B3)→25 (B3|surplus value) / 25 (capitalist|B3) →25 (B3|surplus value)[8]
8) 50 (B2|surplus value) / 50 (capitalist|B3)→50 (B3|B2)→50 (B2|B1)

→50 (B1|surplus value)/50 (capitalist|B3) →50 (B3|B2) →50 (B2|surplus value)[9]

Finally, the remaining four segments belong to the third group, i.e. a set of segments which could be called ‘internal replacement’-type of circulation. In each segment of this type, the transactions are carried out within each production sector. There needs not to be any intersection with another sector, so that the money does not flow out of the sector, but takes the following routes:

9) 100 (A2|A2)→100 (A2|A2)[10]
10) 100 (A1|A1)→100 (A1|A1) →100 (A1|A1)[11]
11) 100 (B2|B2)→100 (B2|B2)[12]
12) 100 (B1|B1)→100 (B1|B1) →100 (B1|B1)[13]

By summing up his investigation on the monetary circuit, Marx concluded with the following propositions:

Propositions of Marx
1) Money advanced by capitalists as wage payments to workers in each sector returns to its starting point[14].
2) Money advanced by capitalists as wage realizes, in the course of its reflux, various components of output, i.e. not only the commodity labour power but also variable-capital part, constant-capital part and surplus value of social products[15].
3) However, some rest (‘Überschuß’) remains which cannot be realized by circulation of money initially advanced as wage payments. In order to realize the rest, it is necessary for capitalists to advance additional money[16].

2.2. Marx’s Method of segmentation
According to Marx’s numerical example, not only wage but also all components of money advanced by capitalists return actually to their starting points. However, it is just on the reflux of wage paid by capitalists to workers in each sector that he concentrated his attention. In contrast to wage payments, another source of monetary circuit, e.g. money expended for capitalists’ individual consumption, was said not to have any inherent cause of reflux in itself[17]. This might be, as we will specify later, an idea pregnant with profound meaning. Furthermore, the idea that money advanced initially as wage is able to realize profits (surplus value) could not be acquired in such 2-sector models as in the edited Volume II of *Capital*. For example, in the sixth segment above, the money returns to its

starting point after realizing the profit of sector A1 and repeating circulation seven times in all.

After all useful results Marx achieved by his investigation on the monetary circuit, it must be said that his method of investigation, i.e. segmentation of economy-wide transactions into independent parts, cannot be applied in general to the issues of monetary circulation. This method of segmentation was initially defined by Marx in Chapter 7 of Volume I of *Capital* where the components of *value* of products could be represented by the components of *quantity* of products. Marx, however, admitted at the time that this method of perception 'can be accompanied by very barbaric ideas' (MEW 23, S.237). Marx regarded Senior's 'last hour' as their representative. According to this segmenting method, a part of products representing surplus value is fictitiously thought to be produced by means of surplus labour only without any assistance of means of production. The fact is, however, that even every infinitesimal part of products contains all components of value, i.e. constant-capital part, variable-capital part and surplus value in itself.

Correspondingly, Marx changed a text in Ms. II which was designated to 'circuit of commodity capital' in *Capital*, Volume II, Chapter 3. In the later Ms. V (1876-77), he added a precautionary remark about that segmenting method: 'In fact (In der That), just as each element of c, v, m existing in thread is divided again into the same components, ... each individual pound of thread can be divided into c)..., v)...,m)...' (MEGAII/11,S.627-8. MEW 23,S.237). Applied to Marx's model of monetary circuit, this suggestion would mean that the proceeds from the sale of products of value of 50 do not replace e.g. wage payments exclusively, but 2/3, 1/6 and 1/6 of them replace the value of means of production, wage and surplus value respectively. There can be no rational reason why one should fix a part of products which is physically indistinguishable from other parts of them to a certain (arbitrary) segment of commodity transactions. In other words, what one should purchase with proceeds of his product, does not depend on to whom one has sold it. It depends on the composition of its cost. For this reason, we should replace Marx's original method of segmentation with a more general one suggested in Ms. V. Then, it is necessary to examine whether Marx's propositions in Section 2.1 must be also rejected along with his segmenting method or they can be fruitfully reconstructed. The next section will deal with this question.

3. Theoretical implications of Marx's model of monetary circuit

3.1. Generalization of Marx's model of monetary circuit

While we maintain some original properties, i.e. that value added consists of wage and profit, and the income of workers and capitalists come respectively from wage and profit exclusively, we generalize the model in the following way:

- The economy consists of n production sectors, workers and capitalists.
-Capitalists may outlay money not only for consumption but also for net investment.
-Concerning the prices, the following is assumed:

(A.1) An arbitrary consistent price system is valid, and the quantity unit of each good is determined so that its price is unity.

Furthermore, the following symbols are introduced.

Symbols

- a_{ij} $(i, j = 1,\ldots,n) \in \mathbb{R}_+$: input coefficient of good i for sector j, i.e. quantity of good i necessary to produce one unit of good j.
- w_j $(j = 1,\ldots,n) \in \mathbb{R}_+$: wage payments per unit of good j.
- π_j $(j = 1,\ldots,n)$ $\in \mathbb{R}_+$: profit per unit of good j.
- b_{il} $(i= 1,\ldots,n) \in \mathbb{R}_+$: consumption coefficient of good i for workers, i.e. quantity of good i consumed (purchased) by workers per income unit.
・ b_{ik} $(i= 1,\ldots,n) \in \mathbb{R}_+$: quantity of good i purchased by capitalists per income unit. In the case of net investment (extended reproduction), production goods are purchased and wage is paid by capitalists' income (surplus value).
- x_j $(j = 1,\ldots,n, l, k)$ $\in \mathbb{R}_+$: activity for sector j, i.e. $x_1, \ldots, x_n$ are the output of each sector, and x_l, x_k are the income of workers and capitalists respectively.
- $x := (x_1, \ldots, x_n, x_l, x_k)$: activity vector
- X $\in \mathrm{M}((n+2)\times(n+2), \mathbb{R}_+)$: diagonal matrix with components of x being its diagonal elements.

- $$A = \{\alpha_{ij}\} := \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_{1l} & b_{1k} \\ \vdots & & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_{nl} & b_{nk} \\ w_1 & \cdots & w_n & 0 & w_k \\ \pi_1 & \cdots & \pi_n & 0 & 0 \end{pmatrix}$$: 'transaction matrix'

- m_{ij} $(i = 1,\ldots,n,w,\pi;\ j = 1,\ldots,n,l,k)$ $\in \mathbb{R}_+$: quantity of money advanced by sector j for purchasing good i.

• $\mu_{ij} = m_{ij}/x_j$ ($i = 1,\dots,n,w,\pi$; $j = 1,\dots,n,l,k$) $\in \mathbb{R}_+$: 'money advance coefficient', i.e. ratio of advanced money m_{ij} to output x_j for sector j.
- $M := \{\mu_{ij}\} \in \mathrm{M}((n+2)\times(n+2), \mathbb{R}_+)$: 'money advance matrix'.

Finally, we make the following assumptions:

(A.2) Activity equilibrium of reproduction is realized, i.e. $x = Ax$
(A.3) The transaction matrix A is indecomposable[18].
(A.4) $\mu_{ij} \leq \alpha_{ij}$ and therefore $M \leq A$, i.e. more money is not advanced than currently needed for purchase.

3.2. Reconstruction of the law of monetary reflux

In the real economy, money is used for purchasing products, paying wage and profit. Leontief and Brody showed that the input-output table as well as the transaction matrix A can be used effectively to analyse money inflow and outflow ('money flow computation') (Leontief/Brody, 1993). We would like to show that Marx's model of monetary circuit has a suitable framework to which the money flow computation of I-O analysis can be effectively applied. Furthermore, Marx's original propositions on monetary circuit (see section 2.1) are to be rationally reconstructed by replacing his original method of segmentation by this money flow computation.

The state of money advance at the initial point is expressed by MX ($\neq 0$) according to the definition. The state of money inflow after commodity transactions can be expressed by using the transaction matrix A according to Leontief and Brody(1993). α_{ij}, i.e. i-th row and j-th column of A denotes that good i is demanded and to be purchased by sector j, and that, from the viewpoint of money, sector j has money to spend on good i (pay to sector i). The state of money inflow after the first circulation is therefore expressed by $A\overline{Mx}$ where for any vector z, $\overline{z}$ is defined as the diagonal matrix with components of z being its diagonal elements. If i-th row and j-th column of $A\overline{Mx}$ is positive, sector j received, as a result of the first circulation, money which is to be spent on good i. In this sense, the transaction matrix A has a meaning of money inflow matrix.

On the other hand, it is possible that not all money received is actually spent. We introduce now a parameter $\beta_{ij} \in [0,1]$ to define the money outflow matrix $B := \{\beta_{ij}\alpha_{ij}\}$. In contrast to money inflow $A\overline{Mx}$ as result of the previous transactions, the state of money outflow as cause of next transactions is expressed by $B\overline{Mx}$. Then, $A - B$ denotes not spending money received and holding money in each sector. This may be interpreted as 'money stock' matrix of Leontief and Brody, which represents money stock to be held in hand per a dollar received. Defining $t_{ij} := 1\text{-}\beta_{ij}$, we obtain $A - B = \{(1 - \beta_{ij})\alpha_{ij}\} = \{t_{ij}\alpha_{ij}\}$ where Leontief and Brody define t_{ij} as 'delay' or 'turnover period' and its inverse as 'velocity' of money (according to their definition, $t_{ij} = 0$ means the infinite velocity of money) (Leontief/Brody, 1993, p.228-9). After the second circulation, money inflow and outflow are expressed then as $AB\overline{Mx}$ and $BB\overline{Mx}$ respectively.

We apply the money flow computation of Leontief and Brody to Marx's (generalized) model of monetary circuit. In the case of $m_{ij} > 0$, i.e. if sector j advanced some money for purchasing good i at the initial point, a proportional part of receipts is withdrawn for recovering the initial money advance every time sector j received money to be spent on good i. From $A - B$, i.e. money receipts that is not to be spent, M is withdrawn for recovering the initial money advance, i.e. $(A-B) \geq M$.
In other words, money to be spent for next transactions, B, can never excess the difference between receipts and recovery, $A - M$, i.e. $(A-M) \geq B$.

As the first and simple case, we consider the case of $M = A - B$ in this section, i.e. the case where after withdrawal for recovering money advance, all remaining money is spent. For a series of circulation, the state of money flow can be calculated as Table 2 shows. The numbers in the 'circulation'-row mean how many times the circulation, i.e. transactions mediated by money, has been carried out. Inflow represents how much money sector j received to spend it on good i as result of the previous transactions. Outflow represents how much money sector j is to spend on good i as cause of next transactions. Recovery stands for how much money sector j withdraws from receipts for recovering its initial money advance. Then, we establish the following proposition.

Proposition 1 (see the proof in Appendix 1)
If $M = A - B \neq 0$ is satisfied,
(i) Either all or asymptotically all output is sold.
(ii) Either all or asymptotically all money initially advanced returns to its starting point.
(iii) There is no money stock formation.

$A - B = M$ means $(1 - \beta_{ij})\alpha_{ij} = t_{ij}\alpha_{ij} = m_{ij}$ and, by using the 'velocity (v_{ij})' of Leontief and

Table 2. Money flow computation

circulation	0(initial point)	1	2	…	n	…	total
inflow		$A\overline{Mx}$	$A\overline{BMx}$		$A\overline{B^{n-1}Mx}$		$A\overline{(I-B)^{-1}Mx}$
outflow	MX	$B\overline{Mx}$	$B\overline{BMx}$		$B\overline{B^{n-1}Mx}$		$MX+B\overline{(I-B)^{-1}Mx}$
recovery		$M\,\overline{Mx}$	$M\,\overline{BMx}$		$M\,\overline{B^{n-1}Mx}$		$M\,\overline{(I-B)^{-1}Mx}$
money stock		$(A-M-B)\overline{Mx}$	$(A-M-B)\overline{BMx}$		$(A-M-B)\overline{B^{n-1}Mx}$		$(A-M-B)\overline{(I-B)^{-1}Mx}$

Brody, $\alpha_{ij} = v_{ij}m_{ij}$. Considering that unit price for every good is unity, the latter formula means price × amount of transactions = velocity of money × quantity of money, which can be interpreted as 'Fisher equation of exchange' on the level of input coefficients just as Leontief and Brody also pointed out (Leontief/Brody, 1993, 229). Proposition 1 claims that if this equation is valid, (i) demand (the row sum of total outflow in Table 2) converges to output x, and (ii) the total recovery converges to the initial money advance MX. We see that Marx's original propositions 1) and 2) on the monetary reflux (see section 2.1) can be thus rationally reconstructed. In fact, Marx's model of monetary circuit is a particular case of the general model presented above, and his numerical example can be expressed in the general framework as follows:

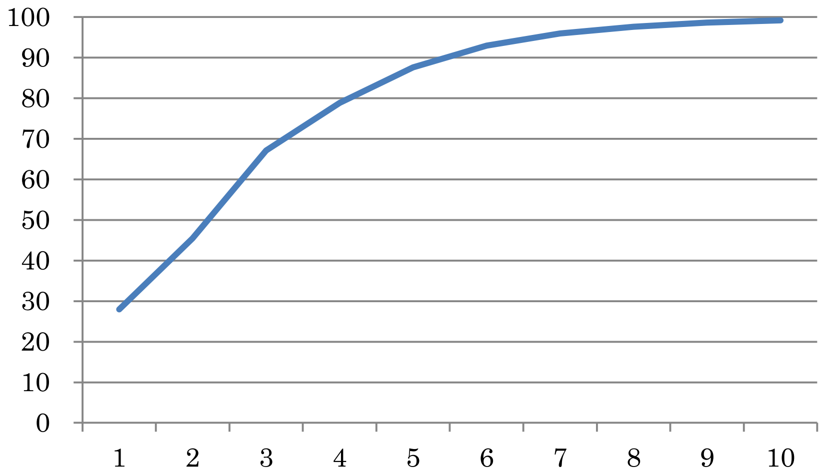
$$A_0 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \end{pmatrix} M_0 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{12} & 0 & 0 \end{pmatrix} B_0 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{12} & 0 & 0 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} 300 \\ 300 \\ 300 \\ 300 \\ 300 \\ 300 \\ 300 \\ 300 \end{pmatrix} \qquad M_0X_0 = \begin{pmatrix} 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 50 & 50 & 50 & 50 & 50 & 50 & 0 & 0 \\ 0 & 0 & 0 & 0 & 50 & 25 & 0 & 0 \end{pmatrix}$$

In Marx's example, the transaction matrix A_0 is indecomposable (cf. (A.3)), and the activity x_0 satisfies the equilibrium condition $x_0 = A_0x_0$ (cf. (A.2)). (A.1) and (A.4) are also assumed. Above all, $M_0 = A_0 - B_0$ is assumed. According to Proposition 1, the initially advanced money M_0X_0 returns gradually to its starting point as transactions are repeated infinitely.

As Marx pointed out, A-sectors have priority over B-sectors in monetary reflux, so that sector A3 finishes its reflux of initial advance already after two times (wage payment by capitalists to workers in the first and purchase of consumption goods by workers in the second circulation), and sector A2 finishes its reflux after the third circulation (purchase of A2 by sector A3). Figure 1 depicts the percentage rate of total recovery to the money advance for the whole economy from the first to the tenth circulation. As we can see, the percentage of recovery excess 90 % after the sixth circulation increasing asymptotically toward 100%.

Figure 1. Rate of monetary reflux (A_0,B_0,M_0,x_0)



3.3. Necessity of additional money advance

We examine next whether Marx's proposition 3) on monetary reflux (section 2.1) can be rationally reconstructed. According to Marx, money is initially advanced by capitalists as wage to workers in each sector. It returns to its starting points after realizing various components of output. However, some rest ("Überschuß") remains which cannot be realized by circulation of money initially advanced as wage payments, and in order to realize this rest, he considered it necessary for capitalists to advance additional money.

First, as we have seen in Proposition 1, as far as $M = A - B$ is satisfied, advanced money returns to its starting point after exhaustively realizing all components of social output x however small the advance may be and wherever it may be advanced (i.e. no matter whether it may be advanced for purchasing some good or for paying wage or profit). Now, maintain A_0 and x_0 and limit initial money advance to wage payments in each sector, i.e. let the money advance matrix be the following.

$$M_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, there can be no rest in system (A_0,B_1,M_1,x_0) as far as $B_1 := A_0 - M_1$ is satisfied. If so, how could one understand rationally Marx's proposition that there remains such rest, and for their realization, money advance in addition to initial wage payments is necessary?

We have so far considered the special case of $M = A - B$. In the case of $A - M \gneq B$ (i.e. $A-M \geq B$ and $A-M \neq B$), however, after withdrawal for recovering money advance, not all remaining money is spent. In this case, new formation of money stock of $(A - M - B)\overline{Mx}$ takes place. Then, the state of money inflow after the second circulation will be $A\overline{BMx}$. After withdrawal for recovering the advance, money outflow for the next time of circulation will be $B\overline{BMx}$, so that money stock formation of $(A - M - B)\overline{BMx}$ will newly take place. As transactions are repeated infinitely, the money flow can be calculated as before (see Table 2)

Proposition 2 (see the proof in Appendix 1)

In the case of $A - B \gneq M \gneq 0$,

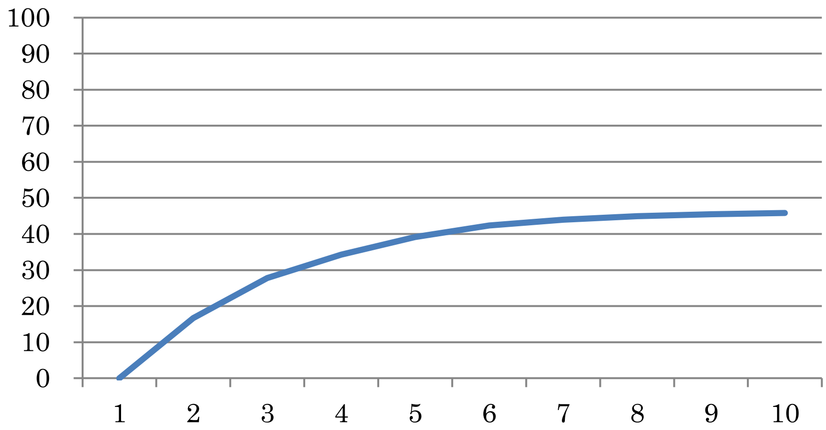
(i) A part of output remains unsold.
(ii) Measured in the aggregate (scalar), the sum of total recovery and total money stock converges to the initial money advance.
(iii) A part of advanced money does not return to its starting point (the total recovery falls short of money advance), i.e. the total money stock is positive.
(iv) Let β_{ij} and m_{ij} be given for all i and j. With $(A - B - M)X$ being advanced additionally to MX, the initially advanced money MX returns to its starting point, and all output x can be sold.

In other words, $A - B \gneq M$ means that there is a pair (i, j) with $(1 - \beta_{ij})\alpha_{ij} = t_{ij}\alpha_{ij} > m_{ij}$ and therefore $\alpha_{ij} > v_{ij}m_{ij}$ so that 'Fisher equation of exchange' on the input-coefficient level is not valid. In this case, in spite of the equilibrium activity (x=Ax), the equality of supply and demand is violated, some output remains unsold, and some money advanced does not relux. To say more formally in view of Table 2, demand (the row sum of total outflow $Mx + B(I - B)^{-1}Mx$) falls short of output x, and the total recovery ($\overline{M(I - B)^{-1}Mx}$) falls short of the initial money advance MX.

Now, let's apply the above results to Marx's example. Maintain the inflow matrix A_0, the outflow matrix B_0, and the activity x_0, and limit initial money advance to wage payments in each sector, i.e. let the money advance matrix be M_1. Then, we have $A_0 - M_1 \gneq A_0 - M_0 = B_0$, and in the system (A_0,B_0,M_1,x_0), new formation of money stock takes place. In this system, monetary reflux takes a quite different course from the system (A_0,B_0,M_0,x_0) or (A_0,B_1,M_1,x_0).

In the system (A_0,B_0,M_1,x_0), although monetary reflux proceeds from A-sectors to B-sectors in the same way as in the system (A_0,B_0,M_0,x_0), a large part of advanced money does not return (see Figure 2).

Figure 2. Rate of monetary reflux (A_0,B_0,M_1,x_0)



300 units of money that were initially advanced realize only 1248.39 units of total output of 1800 units, so that 551.61 units of output remain unsold. And only 138.71 of total money advance of 300 return, so that the rest is hold as money stock in the middle of its

circulation. The default in realization and reflux results from the ‘delay (turnover period)’

of circulation that amounts to the following in the case of the outflow matrix B_0: $t_{22} = 1 - \beta_{22} = 1$; $t_{55} = 1 - \beta_{55} = 1$; $t_{11} = 1 - \beta_{11} = 1/2$; $t_{44} = 1 - \beta_{44} = 1/2$; $t_{68} = 1 - \beta_{68} = 1/2$; $t_{58} = 1 - \beta_{58} = 1$. Because of this delay, new formation of money stock takes place that amounts to 66.67 in sector A2, 33.33 in sector A1, 9.68 in sector B3, 38.71 in sector B2 and 12.90 in sector B1. In practice, the decrease of β_{ij}, i.e. increase of t_{ij} is, inter alia, due to
- the presence of money stock necessary for time profile of 'point-input and continuous output' (e.g. fixed capital or stock of circulating capital); and/or
- the presence of money stock as accumulated profit.

Once we assume an excess delay (i.e. $t_{ij}\alpha_{ij} > m_{ij}$), additional money advance is in fact necessary to realize the total output x_0 and to recover the initial money advance M_1X_0 as suggested by Marx's proposition 3) (section 2.1). In particular, Marx's additional advance $(A_0 - B_0 - M_1)X_0$ would be able to do this task justice. However, it is notable that in contrast to the initial advance, the additionally advanced money would not reflux in the strict sense of the word. It would end in being held as money stock in those sectors who has the excess delay. In Marx's example, it would 'reflux' to the same sector, but not to the same capitalist, it would move from investors to savers within the sector.

In this sense, the initial money advance M_1X_0 and the additional one $(A_0 - B_0 - M_1)X_0$ are quite different in their theoretical quality. As mentioned in Section 2.2, Marx stressed the difference of wage payments from another source of monetary circuit in the omitted part of Ms.II. It is possible and meaningful to make sense of his suggestion by comprehending this qualitative difference. In doing so, we could reevaluate Marx's own contribution especially to the theory of monetary circuit along the line of the so-called 'Franco-Italian circuit school', e.g. A. Graziani.

Graziani makes a categorial distinction between 'initial finance' and 'final finance'. The former is used, just as Marx's initial advance, for wage payment to workers by capitalists and 'in order to bridge the gap between production and resale of commodities', and it is therefore 'in the nature of temporary finance'. On the other hand, 'final finance' is needed when 'initial finance' cannot be fully recovered. 'It would be wrong to think that the cost of consumer goods should be totally covered by proceeds from the sales of consumption goods... What matters to firms is that final finance be sufficient to cover total initial finance' (Graziani, 1990, 394-5). Graziani's distinction of initial and final finance is essential to his theory of endogenous money in the sense that for initial finance, credit money is created by the bank while final finance is supplied by new issues of securities and stocks.

4. Conclusion

It is true that Marx's model of monetary circuit in Ms. II is not explicitly based on bank credit and bank notes, which play a central role in the circuit school. However, as Graziani acknowledged, as far as inter- (and intra-)sectoral monetary circulation ('circulation phase') is concerned, it does not matter whether commodity money or credit money is considered (Graziani, 1998, 32-3). Despite this difference, Marx’s six-sector model of monetary circuit anticipated *de facto* the main results of the so-called 'Franco-Italian circuit school' in that Marx acknowledged, just as e.g. A. Graziani (1990, 1998, 2003), E. J. Nell (2004) and others, the wage payments to be the main source of the monetary circulation. Now that the discussion of the circuit school revolves mainly around the macro-level monetary circulation between capitalists and workers ('initial phase'), a rational reconstruction of Marxian 6-sector model may be useful as a clue to promote a multi-sectoral analysis of monetary circuit. Furthermore, Marx’s multi-sectoral model of monetary circuit provides a suitable framework to which we can effectively apply the money flow computation of I-O analysis by Leontief and Brody (1993). As stated in Introduction, Marx’s 6-sector model was systematically deleted by Engels when he edited *Capital*, Volume II. If this study had appeared in the *Capital* edition, it might have perhaps lead readers to further develop his multi-sectoral model of monetary circuit.

Concerning Marx's own process of research, money stock formation as amortization funds and accumulation funds was to be explicitly treated in the framework of reproduction schemata for the first time in the later Ms. VIII. However, although Marx did not explicitly account for fixed capital and accumulation in the omitted part of Ms. II, his reasoning would consequently require to assume their existence. In this sense, the treatments in Ms. VIII, although they remain incomplete and partly defective[19], ware already inevitable.

Appendix 1

Proof of Propostion 1

The proposition follows from $x = Ax$ and Table 2. First, $x = Ax$ follows from Assumption (A.2). Then, $x = (A - M)x + Mx$ follows. Since A is indecomposable and because of $A \geq M$, $M \neq 0$, the Frobenius root of the non-negative matrix $(A - M)$ is less than unity. Therefore, we obtain

$x = (I - A + M)^{-1}Mx = (I - B)^{-1}Mx = (I + B + B^2 + \ldots)Mx$. From this, (i) the total outflow (demand) is X, (ii) the total recovery is MX, and (iii) the total money stock is zero because $A-B-M = 0$. *qed*

Proof of Proposition 2

Concerning the total value of inflow, outflow, recovery and money stock, we refer to Table 2.

(i) Because of $A - M \gneq B$, $M \neq 0$ and the indecomposablity of A, the Frobenius root of the non-negative matrix B is less than unity. From (A.2) follows $x = Ax$ with $x > 0$. Then, we have

$$x = Ax - Bx - Mx + Mx + Bx = (I - B)^{-1}(A - B - M)x + (I - B)^{-1}Mx \gneq (I - B)^{-1}Mx$$

and therefore

$$x = Ax \gneq (M + B)x \geq Mx + B(I - B)^{-1}Mx .$$

(ii) Define $\iota := (1, ..., 1)$. Aggregating the total recovery and the total money stock, and considering unit price of each good being unity, we obtain

$\iota(A - B)(I - B)^{-1}Mx = \iota(I - B)(I - B)^{-1}Mx = \iota Mx$.

(iii) It is $MX \gneq M\overline{(I - B)^{-1}Mx}$ that should be proved. Obviously,

$D := Mx - M(I - B)^{-1}Mx = M(I - B)^{-1}(x - Bx - Mx) = M(I - B)^{-1}(A - B - M)x$

is valid. We examine now the structure of the matrix $M(I - B)^{-1}(A - B - M)$.

Let S be the set of the indices of those columns of M that contain a positive element, and S^C be the complement of S. Because of $M \gneq 0$, $S \neq \varnothing$. If $S^C = \varnothing$, then $D \gneq 0$ because of $(A - B - M) \gneq 0$ and $x > 0$. Therefore, we need to consider only the case of $S^C \neq \varnothing$. Let

$[(I-B)^{-1}]_{ij}$ be (i, j)-element of $(I-B)^{-1}$. Define T so that $T := \{j \mid \forall i \in S : [(I-B)^{-1}]_{ij} = 0\}$.

Let T^C be the complement of T. Since all diagonal elements of $(I-B)^{-1}$ are positive, we have $T \subset S^C$ and therefore $S \subset T^C$. If $T = \emptyset$, then $D \gneq 0$ because of $(A - B - M) \gneq 0$ and $x > 0$. Therefore, we need to consider only the case of $T \neq \emptyset$.

We assume now $D = 0$. Then, those rows of $(A - B - M)$ whose index belongs to T^C must be null. According to the definition of S and T, (i, j)-element of $(I-B)^{-1}$ is null for all $(i, j) \in S \times T$. Then, the (i, j)-element of $(I-B)^{-1}$ is null also for all $(i, j) \in T^C \times T$ because if the (i, j)-element of $(I-B)^{-1}$ was positive for some $(i, j) \in (T^C - S) \times T$, for this (i, j) and some $k \in \mathbb{N}$, the (i, j)-element of B^k would be positive (consider $(I-B)^{-1} = I + B + B^2 + \ldots$). Then, for some $(i, j) \in S \times T$, the (i, j)-element of B^{2k} as well as $(I-B)^{-1}$ would be positive, and this would contradict the definition of T. Therefore, the (i, j)-element of $(I-B)^{-1}$ as well as B must be null for all $(i, j) \in T^C \times T$. On the other hand, according to the definition of S, the (i, j)-element of M is null for all $(i, j) \in T^C \times S^C$. Because of $T \subset S^C$, the (i, j)-element of $(B + M)$ is null for all $(i, j) \in T^C \times T$. However, as stated above, $D = 0$ implies that those rows of $(A - B - M)$ whose index belongs to T^C must be null. Therefore, the (i, j)-element of A must be null for all $(i, j) \in T^C \times T$. This contradicts the indecomposability of A. Therefore, $D \gneq 0$.

(iv) Considering $x = Ax - Bx + Bx = (I-B)^{-1}(A - B)x$, the demand (the row sum of total outflow) will increase by additional money advance to the following amount.

$$(M + A - B - M)x + B(I-B)^{-1}(M + A - B - M)x$$
$$= (A-B)x + B(I-B)^{-1}(A-B)x$$
$$= Ax = x$$

Furthermore, we will obtain the following recovery.

$$M\overline{(I-B)^{-1}(M + A - B - M)x}$$
$$= MX$$

qed.

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Footnotes

[1] Marx used in his original awkward notations IIα, IIαα, Ia, IIβ, IIββ and Ib, which we substitute in this paper with A1, A2, A3, B1, B2 and B3 respectively.

[2] MEGAII/11,S.444,453.

[3] MEGAII/11,S.444,446,455.

[4] MEGAII/11,S.445-7,456.

[5] MEGAII/11,S.444,453-4.

[6] MEGAII/11,S.445,447,456,476.

[7] MEGAII/11,S.448,459,476.

[8] MEGAII/11,S.444,455.

[9] MEGAII/11,S.450-1,460,476.

[10] MEGAII/11, S.449, 459.

[11] MEGAII/11, S.449, 459.

[12] MEGAII/11, S.451, 460.

[13] MEGAII/11, S.451, 460.

[14] 'The money of 300₤ that is initially advanced in the class I(Ia and Ib), (IIαα + IIα) and (IIββ + IIβ) as variable capital and then spent by workers as means of purchase , means of circulation of their revenu on necessary means of consumption, has returned everywhere to its starting point in order to begin then with its course anew'. (MEGAII/11,S.469)

[15] 'The same amount of money of 50₤ figures here alternately as money form of variable capital (IIβ)..., then money form of labours' revenue..., money form of constant capital (Ia)..., money form of constant capital (IIαα)..., money form of capitalists' revenue (IIα)..., money form of constant capital (Ib)..., money form of constant capital

(IIββ)..., and finally returning money form of variable capital (IIβ)'. (MEGAII/11,S.463-4)

[16] 'It is therefore the variable money capital that mediates all these transactions. However, there remains as rest (Ueberschuß) a part of surplus value = I(b), II(ββ + β) which is not circulated by it, and for whose circulation own amount of money must be advanced'. (MEGAII/11,S,491) ‘The sum of value of total circulation is divided into two independent parts of which the one is advanced for circulating variable capital and surplus labour, and the other circulates within constant capital reproduced in their various natural forms. It is a big mistake to think that the amount of money outlaid by consumers (i.e. wage + capitalists' revenue) purchases and circulates the whole product'. (MEGAII/11,S.481)

[17] 'The capitalist who spends money for purchasing his means of consumption does not cause the reflux of this money to his pocket by this act as itself. ... It returns to him ... because he realizes, by the sale of his commodity, the surplus value contained in his commodity. That surplus value is contained in his commodity, and that therefore surplus value is realized by the sale thereof, has absolutely nothing to do with it and does not follow from it that he spent money for purchasing means of consumption'. (MEGAII/11,S.462-3)

[18] This assumption is satisfied particularly if the n-dimensional principal submatrix of $A,\{a_{ij}\}(i, j = 1, ..., n)$, is indecomposable and $(b_{1l},...,b_{nl})',(b_{1k},...,b_{nk})',(w_1,...,w_n)$ and $(\pi_1,...,\pi_n)$ are each semi-positive.

[19] Regarding Marx's confusion about schemata of extended reproduction, see MEGAII/13, Apparat, Einführung.